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Three-Dimensional Flow Theory of Turbomachinery, Part 1: Basic Methodology

Jianzhong Xu,* Manchu Ge,† Xiaolu Zhao,† and Zhengming Wang†
Chinese Academy of Sciences, 100080 Beijing, People's Republic of China

This paper is the first part of a two-part paper (see "Three-Dimensional Flow Theory of Turbomachinery, Part 2: Design and Analysis System," *Journal of Propulsion and Power*, Vol. 14, No. 6, 1998, pp. 907–915). This paper is concerned with the basic concept and methodology of the three-dimensional flow theory of turbomachinery and the solutions of rotational flow in subsonic and transonic turbomachines. In this paper, the fundamental assumptions and the governing equations used in the theory are described briefly. Some discussions are given about the quintessence of this theory: the partial derivatives along a stream surface and the thickness of the stream filament as well as the blade force between the stream surfaces appeared in the principal equations of two-dimensional flow on one kind of stream surface. A brief review of the methods associated with the solutions of the flow along S_1 and S_2 stream surfaces in subsonic and transonic turbomachines are carried out. Some emphases are placed on a body-fitted non-orthogonal curvilinear coordinate system and the solution of a transonic stream function equation.

Nomenclature

$A_1, A_2, A_3, A_4, A_5, A_6$ = coefficient in dynamic equation
a_{ij} = covariant component of two-dimensional metric tensor
ds = line length
$\boldsymbol{e}$ = vector
e^i = contravariant component of $\boldsymbol{e}$
F = force acting on S_2 surface per unit mass of fluid
g^{jk} = contravariant component of three-dimensional metric tensor
g_{jk} = covariant component of three-dimensional metric tensor
H = stagnation enthalpy
I = stagnation rothalpy
l = mixing length or generator of the surface of revolution
l, φ = orthogonal coordinates on surface of revolution
M = Mach number
$\boldsymbol{n}$ = unit vector normal to stream surface
n_i = covariant component of $\boldsymbol{n}$
Pr = Prandtl number
p = pressure
q = any fluid quantity
R = gas constant
r, φ, z = relative cylinder coordinates
s = entropy
t = time
u = tangential velocity component
$\boldsymbol{V}$ = absolute velocity vector
$V_\theta r$ = angular momentem of fluid about axis of rotation
$\boldsymbol{W}$ = relative velocity vector
w^i = contravariant component of $\boldsymbol{W}$
w_i = covariant component of $\boldsymbol{W}$
x^i = arbitrary curvilinear coordinate
z = axial coordinate
ε = dissipation rate
η = coordinate used for calculation
θ = angle between x^1 and x^2 coordinates
κ = specific heat ratio or turbulent kinetic energy
λ = heat conduction coefficient
μ = dynamic viscosity
ν = coefficient of kinetic viscosity
ξ = coordinate used for calculation
π' = stress tensor
ρ = density
σ = angle between the flow and axial direction along meridional plane
τ = viscous stress
$\tilde{\tau}$ = normal distance between two adjacent stream surfaces
Φ = dissipation function
φ = tangential coordinate
ψ = stream function
ω = angular velocity

Subscripts

j = along the x^1 coordinate
k = along the x^2 coordinate
r, φ, z = radial, circumferential, and axial component
θ = absolute tangential direction
φ = relative tangential direction

Superscript

$-$ = on stream surface or dimensionless quantity

I. Introduction

IT is well known that the flows through a turbomachinery are among the most complex flows encountered in aerothermodynamics because of the complicated geometry of turbomachinery, gas viscosity, shocks, tip leakage, rotation, rotor–stator interaction, and flow–solid interactions. For such a

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*Academician, Institute of Engineering Thermophysics, P.O. Box 2706.
†Professor, Institute of Engineering Thermophysics, P.O. Box 2706.

three-dimensional, unsteady flow of a viscous fluid, it was impossible to solve it completely and accurately in the beginning of the 1950s, and even up to the present time.

In the late 1940s, when the first set of electronic computers were invented and used, Wu[1–4] predicted that the digital computational method would be much more powerful than the analytical method for solving the problem of flows in turbomachinery, although the latter played a dominant role in fluid dynamics at that time. Most important was integrating the physical model along with the computational technique. He made some fundamental assumptions that could simplify the complex flow phenomena and help capture the nature of the fluid motion in turbomachines. Wu introduced the S_1 and S_2 families of relative stream surfaces that eliminated three-dimensional flow problems of coupling solutions of two-dimensional flow along two sets of stream surfaces. Based on this concept, quasi-three-dimensional approaches have been developed and widely used for over 30 years, and they are still employed as present day tools for many designers.

A. Fundamental Assumptions of the Three-Dimensional Flow Theory

With regard to the complicated flow in turbomachinery Wu made three essential assumptions and proposed his well-known general theory of three-dimensional flow in turbomachinery.[1–4]

1. *Relative Steady Flow*

The flow in turbomachinery is always unsteady. Both the relative and absolute flows are unsteady in stator and rotor blade rows. As an approximation, the fluid flows through the stators and rotors are assumed to be steady with respect to the stationary and rotating blade rows, respectively.

2. *Approximate Model to Control Gas Viscosity*

The effect of fluid viscosity on the flow is significant and must be incorporated in the three-dimensional flow solution. In the core region of the blade row flow, the viscous stress is negligible in the dynamic equation, but in the region near the solid wall, the viscosity effect on the flow may be involved through the entropy gradient. Usually, the viscous loss distribution from hub-to-tip is included in the mean S_{2m} stream surface equation to define the gross effect of viscosity on the three-dimensional flow. This distribution can be obtained from the empirical loss correlation or from experimental data. With regard to the blockage effect caused by boundary-layer development, the mass flow coefficient is introduced in the continuity equation of the flow along a stream surface.

3. *Adiabatic Flow*

Because the flow path in each blade row is not long relatively, and the variation in temperature is not significant, the heat transfer between the flow and its surrounding are small, and the assumption of adiabatic flow is adopted.

B. Governing Equations for Three-Dimensional Flow in Turbomachinery

The basic aerothermodynamic equations governing the three-dimensional flow of a viscous fluid in a relative coordinate system with a constant angular velocity have been used in the general three-dimensional flow theory[1–3]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{W}) = 0$$

$$\frac{d\boldsymbol{W}}{dt} - \omega^2 \boldsymbol{r} + 2\boldsymbol{\omega} \times \boldsymbol{W} = -\frac{1}{\rho}\nabla p + \nabla \cdot \boldsymbol{\pi}$$

$$\frac{dI}{dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \dot{q} + \frac{1}{\rho}\nabla \cdot (\boldsymbol{\pi}' \boldsymbol{W})$$

where $\dot{q}$ is the heat added to the fluid per unit mass per unit time. For a steady flow, the continuity equation becomes

$$\nabla \cdot (\rho \boldsymbol{W}) = 0 \tag{1}$$

With the assumptions described earlier, the dynamic equation of a relative steady flow is approximated as follows:

$$\boldsymbol{W} \times (\nabla \times \boldsymbol{V}) \approx \nabla I - T\nabla s \tag{2}$$

In the core region of flow, viscous stress and heat transfer are negligible. In the boundary-layer region near solid walls, if the boundary layer is laminar, the Pr of the fluid is equal to unity, and the assumption of adiabatic walls is adapted, and the viscous work and heat transfer terms cancel each other. In an actual turbomachine, the flow is turbulent and the Pr is different from 1, and the sum of these two terms will not equal zero, but its magnitude is expected to be small. Therefore, a good approximation for the entire flow region can be obtained:

$$\frac{dI}{dt} \approx 0 \tag{3a}$$

The second law of thermodynamics

$$\frac{ds}{dt} \geq 0 \tag{3b}$$

There are two important thermodynamic properties, I and s, in this set of equations. They are of great significance in the three-dimensional flow theory in turbomachinery. Owing to the employment of I, the energy equation is converted to an algebraic equation and the computation consuming is reduced greatly. As predicted in Ref. 4, the variation of I along a streamline is very small, even in a viscous flow.[5] It has also been demonstrated that I is an invariant across a shock wave.[6] The assumption that the rothalpy remains constant along a relative streamline is a valuable approximation. The use of both s and I makes the momentum equation to be in a first-order partial differential equation form, which is easy to deal with in calculation. It should be emphasized that although the local viscous terms are neglected in the momentum equation, the entropy gradient in the right side of the motion equation represents the accumulation influence of viscous.

II. Theory of Two Kinds of Relative Stream Surfaces

To solve the three-dimensional flow in a relatively simple manner, Wu introduced the basic concept of S_1 and S_2 stream surfaces. Considering the three-dimensional flow in the channel between two blades, one can imagine that the whole flowfield consists of innumerable streamlines spread in the blade channel, and these streamlines constitute innumerable stream surfaces according to certain regulations. The first kind of relative stream surfaces is one whose intersection with a z plane upstream of the blade row forms a circular arc (Fig. 1), and the second is one whose intersection with a z-plane forms a radial line (Fig. 2). The first type of blade-to-blade stream surface and the second type of hub-to-tip stream surface were designated as stream surfaces S_1 and S_2. In general, both of these two families of stream surfaces are employed in the solution of three-dimensional flow in turbomachines. The correct solution of one kind of surface requires some data from the other, and iterative solutions between the two kinds of surfaces are needed to obtain the three-dimensional flow solution.

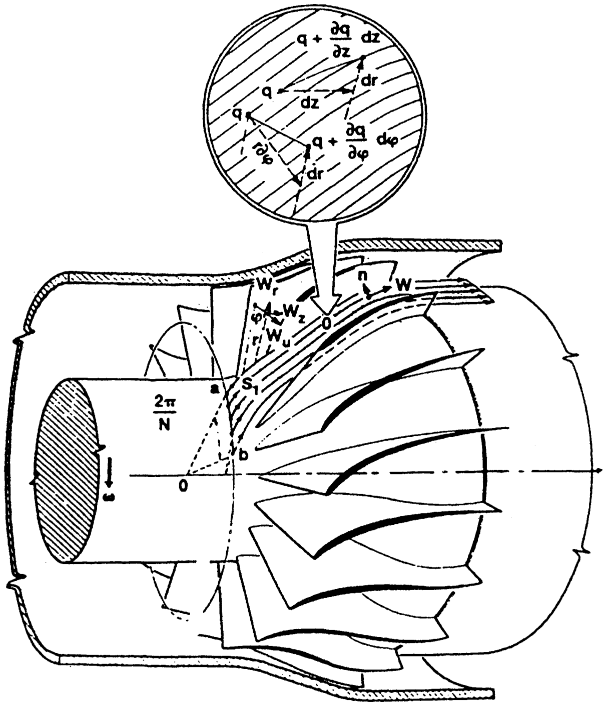

Fig. 1 Relative stream surface S_1.

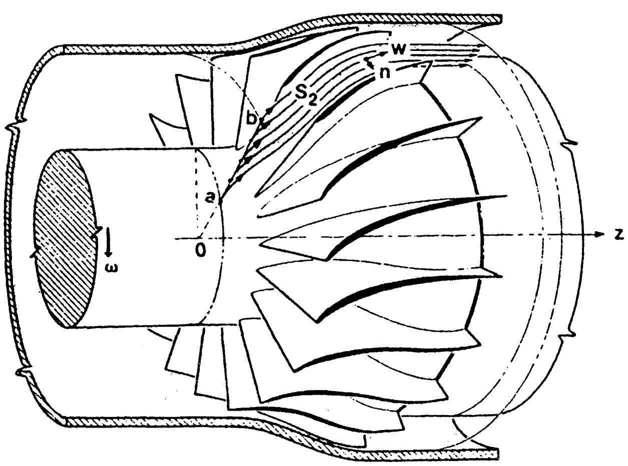

Fig. 2 Relative stream surface S_2.

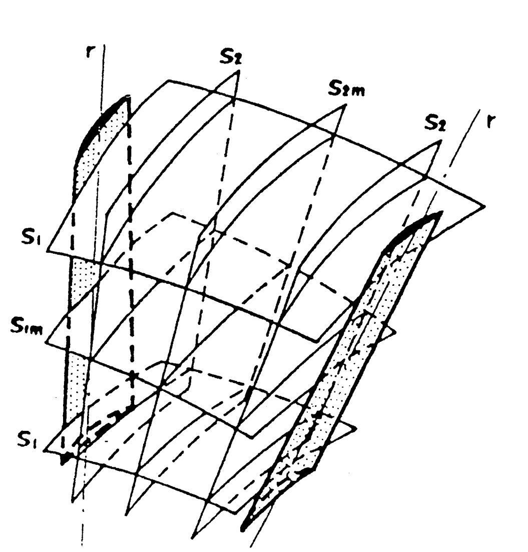

Fig. 3 Intersecting S_1 and S_2 stream surfaces in blade passage.

Based on a nonorthogonal curvilinear coordinate system, a more general expression about two kinds of relative stream surfaces was described in Refs. 7 and 8.

Now, the entire three-dimensional flow is decomposed from two-dimensional flows on the two intersected kinds of stream surfaces (Fig. 3), and the solution of the three-dimensional flow can be obtained from iterations of the solutions of these two types of two-dimensional flows. It is seen that the three-dimensional flow in turbomachinery can be solved by the method of reduction of dimensions, and the numerical solution becomes realizable by the use of different digital computational techniques and hardwires.

A. Equations for Fluid Flow Along S_2 Stream Surface

For the flow along S_2 stream surfaces, Eqs. (1) and (2) are used to eliminate one of the three independent variables: the coordinate ϕ. This means that q on S_2 is now considered as

$$q = f[r, z, \varphi(r, z)] \tag{4}$$

In general, the coordinates of the S_2 stream surface and the components of $\boldsymbol{n}$ (Fig. 2), as well as the velocity components, are related by the following relations:

$$S(r, \varphi, z) = 0 \tag{5}$$

$$n_r dr + n_\varphi d\varphi + n_z dz = 0 \tag{6}$$

$$n_r W_r + n_\varphi W_\varphi + n_z W_z = 0 \tag{7}$$

From Eqs. (4) and (6) the partial derivatives along the stream surface can be obtained

$$\frac{\bar{\partial} q}{\bar{\partial} r} = \frac{\partial q}{\partial r} - \frac{n_r}{n_\varphi}\frac{1}{r}\frac{\partial q}{\partial \varphi} \tag{8a}$$

$$\frac{\bar{\partial} q}{\bar{\partial} z} = \frac{\partial q}{\partial z} - \frac{n_z}{n_\varphi}\frac{1}{r}\frac{\partial q}{\partial \varphi} \tag{8b}$$

Figure 4 shows such an element of S_2 stream filament, and τ is the circumferential thickness of the element. Now, continuity Eq. (1) can be represented by two independent variants, r and z, in the following form:

$$\frac{\bar{\partial}(\tau \rho W_r)}{\bar{\partial} r} + \frac{\bar{\partial}(\tau \rho W_z)}{\bar{\partial} z} = 0 \tag{9}$$

In actual calculations, only the τ to τ_i ratio is important. In general, it is easier to obtain the variation in ratio τ/τ_i from the distance between adjacent streamlines on the S_1 surface.

Considering the flow along the S_2 stream surface and substituting Eqs. (7) and (8) into Eq. (2), the dynamic equations on the S_2 stream surface are reduced to

$$-\frac{W_\varphi}{r}\frac{\bar{\partial}(V_\theta r)}{\bar{\partial} r} + W_z\left(\frac{\bar{\partial} W_r}{\bar{\partial} z} - \frac{\bar{\partial} W_z}{\bar{\partial} r}\right) = -\frac{\bar{\partial} I}{\bar{\partial} r} + T\frac{\bar{\partial} s}{\bar{\partial} r} + F_r \tag{10a}$$

$$F_\varphi r = \frac{D(V_\theta r)}{Dt} \tag{10b}$$

$$-W_r\left(\frac{\bar{\partial} W_r}{\bar{\partial} z} - \frac{\bar{\partial} W_z}{\bar{\partial} r}\right) - \frac{W_\varphi}{r}\frac{\bar{\partial}(V_\theta r)}{\bar{\partial} z} = -\frac{\bar{\partial} I}{\bar{\partial} z} + T\frac{\bar{\partial} s}{\bar{\partial} z} + F_z \tag{10c}$$

where $\boldsymbol{F}$ is a vector having the unit of force per unit mass of gas defined by

$$\boldsymbol{F} = -\left(\frac{1}{r n_\varphi \rho}\frac{\partial p}{\partial \varphi}\right)\boldsymbol{n}$$

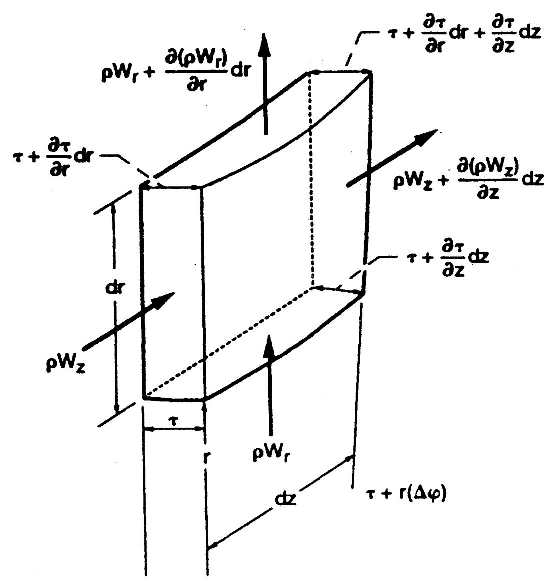

Fig. 4 Element of S_2 stream filament.

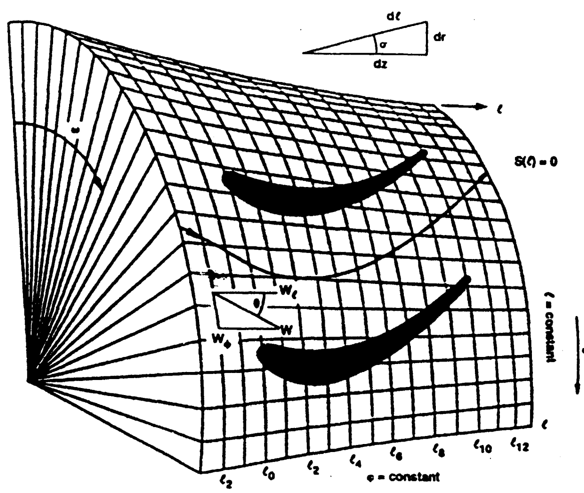

Fig. 5 Orthogonal curvilinear coordinates l and ϕ on S_1 stream surface of revolution.

B. Equations for Fluid Flow Along S_1 Stream Surface

In a quasi-three-dimensional solution, the S_1 stream surface can be a surface of revolution, which is generated by rotating a steamline obtained from an S_{2m} stream surface solution, as shown in Fig. 5. By the use of the orthogonal curvilinear coordinates l and ϕ on the S_1 stream surface of revolution, the equations of continuity and motion are reduced as follows:

$$\frac{\partial(\tau\rho W_l)}{\partial l} + \frac{\partial(\tau\rho W_\varphi)}{\partial\varphi} = 0 \qquad (11)$$

$$\frac{1}{r}\frac{\partial W_l}{\partial\varphi} - \frac{\partial W_\varphi}{\partial l} - \left(\frac{W_\varphi}{r} + 2\omega\right)\sin\sigma = \frac{1}{rW_l}\left(\frac{\partial I}{\partial\varphi} - T\frac{\partial s}{\partial\varphi}\right) \qquad (12)$$

where the τ is the thickness of the S_1 stream filament of revolution in the direction normal to the S_1 surface. Very similar to the S_2 stream surface solution, only τ to τ_i is important, and the variation of τ/τ_i may be obtained from the distance between adjacent streamlines on the S_{2m} surface. For the fully three-dimensional solution, a more general twisted S_1 stream surface is introduced as described in Refs. 9 and 10.

III. General Methods of Solution for Flows on Stream Surfaces

A. Nonorthogonal Curvilinear Coordinate System

It has been seen that the basic equations of motion along S_1 and S_2 stream surfaces are expressed with respect to a body-fitted general nonorthogonal curvilinear coordinate system. These coordinates can adapt to the arbitrarily complicated geometry of turbomachinery and increase the precision of the numerical solution. At the same time, it makes the governing equations of flow on two kinds of stream surfaces of generalized and universalized significance and, thus, allows computer code commonality. The arbitrary nonorthogonal curvilinear coordinate system has been widely used in the flow calculations of turbomachinery, and even in the computation of other engineering problems. This is another significant contribution of Wu to the aerothermodynamics of turbomachinery.

The most notable characteristics seen when applying the nonorthogonal coordinate system to the computation on S_1 and S_2 stream surfaces is the employment of its corresponding nonorthogonal components of velocity. With the aid of this method the three-dimensional velocity is expressed completely, exactly, and naturally; the governing equations of motion are satisfied rigorously; the boundary conditions for the velocity are simplified because one velocity component vanishes; and the overall calculation is simple and accurate.

To solve the governing equations along any kind of stream surface, their three-dimensional coordinates and the thickness of the corresponding stream filaments must be specified and all of these quantities are obtained from the calculation on other kinds of surfaces. It is also necessary to give the streamwise distributions with the same gas parameters at the inlet and exit. For the S_1 computation, the periodicity condition is adopted outside the cascade. In the direct problem the coordinates of the blade on the S_1 surface should be assumed, whereas in the inverse problem the distributions of gas parameters along the cascade surface will be specified and the coordinates of the blade are unknown. Similarly, for the direct problem of the S_2 surface, the flow path on the surface is given and for its inverse problem the distribution of some gas parameters on the inner and outer walls of the flow path are prescribed.

The numerical method to solve the governing equations of flows on the stream surfaces may be divided into two categories. The first approach is the stream function method in which the stream function satisfies the continuity equation and one of the dynamic equations is converted to its principle equation. The introduction of the intermediate variable decreases the number of governing equations and, hence, simplifies the computation.[11–16] The stream function equation may be discretized by means of the central difference or the upwind difference, depending on the flow character, and then the corresponding algebraic equation is obtained and may be solved by means of the matrix relaxation method. The boundary conditions required for solving the equation are easy to specify. The only case that should be treated carefully is the critical condition encountered in the transonic and supersonic flows, and one boundary condition should be dropped.

B. Stream Function Method

Now take the S_2 surface as an example. With respect to the nonorthogonal curvilinear coordinate system on the S_2 surface (Fig. 6), from the continuity equation ψ is introduced:

$$\frac{\partial\psi}{\partial x^2} = \tau\rho w^1\sqrt{a_{22}}\sin\theta_{12} \qquad (13a)$$

$$\frac{\partial\psi}{\partial x^2} = -\tau\rho w^2\sqrt{a_{11}}\sin\theta_{12} \qquad (13b)$$

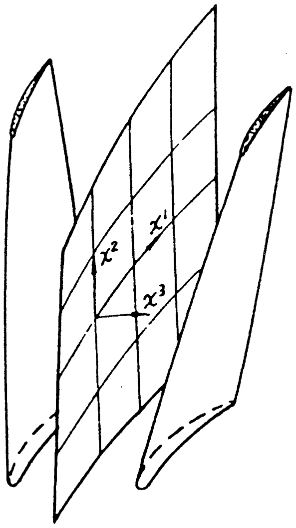

Fig. 6 Nonorthogonal curvilinear coordinates used for S_2 flow.

Substituting the definition of the stream function [Eqs. (13a) and (13b)] into a dynamic equation results in

$$\frac{1}{a_{11}}\frac{\partial^2\psi}{\partial(x^1)^2} - 2\frac{\cos\theta_{12}}{\sqrt{a_{11}a_{22}}}\frac{\partial^2\psi}{\partial x^1\partial x^2} + \frac{1}{a_{22}}\frac{\partial^2\psi}{\partial(x^2)^2} + \frac{J}{\sqrt{a_{11}}}\frac{\partial\psi}{\partial x^1} + \frac{K}{\sqrt{a_{22}}}\frac{\partial\psi}{\partial x^2} = M \tag{14}$$

where

$$J = -\frac{\partial\,\ell n(\tau\sqrt{a_{11}/a_{22}}\sin\theta_{12})}{\sqrt{a_{11}}\partial x^1} + \frac{\cos\theta_{12}}{\sqrt{a_{22}}}\frac{\partial\,\ell n\,\tau}{\partial x^2} + \frac{1}{\sin\theta_{12}\sqrt{a_{22}}}\frac{\partial\theta_{12}}{\partial x^2}$$

$$K = -\frac{\partial\,\ell n(\tau\sqrt{a_{11}/a_{22}}\sin\theta_{12})}{\sqrt{a_{22}}\partial x^2} + \frac{\cos\theta_{12}}{a_{11}}\frac{\partial\,\ell n\,\tau}{\partial x^1} + \frac{1}{\sin\theta_{12}\sqrt{a_{22}}}\frac{\partial\theta_{12}}{\partial x^1}$$

$$M = \left(\frac{1}{\sqrt{a_{11}}}\frac{\partial\,\ell n\,\rho}{\partial x^1} - \frac{\cos\theta_{12}}{\sqrt{a_{22}}}\frac{\partial\,\ell n\,\rho}{\partial x^2}\right)\frac{1}{\sqrt{a_{11}}}\frac{\partial l\psi}{\partial x^1} + \left(\frac{1}{\sqrt{a_{22}}}\frac{\partial\,\ell n\,\rho}{\partial x^2} - \frac{\cos\theta_{12}}{\sqrt{a_{11}}}\frac{\partial\,\ell n\,\rho}{\partial x^1}\right)\frac{1}{\sqrt{a_{22}}}\frac{\partial l\psi}{\partial x^2} + \frac{\tau\sin\theta_{12}}{\sqrt{a_{11}a_{22}}}\rho C$$

$$C = \frac{\sqrt{a_{11}}}{w^1}\left[-\frac{w_\varphi}{r}\frac{\partial(V_\theta r)}{\partial x^2} + \frac{\partial I}{\partial x^2} - T\frac{\partial s}{\partial x^2} - f_2\right]$$

For a three-dimensional design problem, a desirable distribution of the angular momentum $V_\theta r$, rather than the shape of the S_2 surface is usually specified. In this case, the stream function equation [Eq. (14)] remains mathematically elliptic as long as the meridional component of the velocity is less than the local speed of sound.[1,17] This means that when the meridional velocity component is subsonic, the transonic flow problem on the S_2 surface may be solved in a simple manner. Of course, to simulate a shocked flow more precisely, the computer code should be modified by adding more computation stations on both sides of the shock and providing the appropriate abrupt changes across the shock to $V_\theta r$ and τ values in these stations.[8]

In the design problem on the S_2 surface, the component of the force between S_2 surfaces, f_2 needed in solving Eq. (14) is to be computed from the following condition of integrability:

$$\frac{f_2}{f_\varphi} = \left(\frac{f_2}{f_\varphi}\right)_0 + \int_{(x^1)_0}^{x^1}\frac{\partial}{\partial x^2}\left(\frac{f_2}{f_\varphi}\right)\mathrm{d}x^1$$

After the flow parameters are computed, the angular coordinates of the S_2 surface can be determined by the relation

$$\varphi = \varphi_0 + \int_{l_0}^{l}\frac{W_\varphi}{r\sqrt{(w^1)^2 + (w^2)^2 + w^1w^2\cos\theta_{12}}}\,\mathrm{d}l$$

It is seen that the shape of the S_2 surface is obtained at the end of the computation.

For the direct problem on the S_1 stream surface from Eq. (7), the stream function may be defined as

$$\frac{\partial\psi}{\partial x^2} = \tau\rho w^1\sqrt{a_{22}}\sin\theta_{12} \tag{15a}$$

$$\frac{\partial\psi}{\partial x^1} = -\tau\rho w^2\sqrt{a_{11}}\sin\theta_{12} \tag{15b}$$

Substituting Eqs. (16a) and (16b) into the motion equation yields

$$\frac{\partial}{\partial x^2}\left(A_1\frac{1}{\rho}\frac{\partial\psi}{\partial x^2} - A_2\frac{1}{\rho}\frac{\partial\psi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(A_2\frac{1}{\rho}\frac{\partial\psi}{\partial x^2} - A_3\frac{1}{\rho}\frac{\partial\psi}{\partial x^1}\right) = A_4 \tag{16}$$

where

$$A_1 = \sqrt{a_{11}}/(\tau_n\sqrt{a_{22}}\sin\theta_{12}), \qquad A_2 = \cos\theta_{12}/(\tau_n\sin\theta_{12})$$

$$A_3 = \sqrt{a_{22}}/(\tau_n\sqrt{a_{11}}\sin\theta_{12})$$

$$A_4 = -2\omega\sqrt{D(z,\varphi)/D(x^1,x^2)}\cos(\hat{n},\hat{r}) + \frac{\sqrt{a_{11}}}{w^1}\left(\frac{\partial I}{\partial x^2} - T\frac{\partial s}{\partial x^2}\right)$$

Discretizing Eq. (16) leads to a matrix equation, which may be solved by use of the Thompson method: $[M][\psi] = [P]$.

C. Method by Use of Primitive Variables

The second approach used to solve the governing equations on the stream surface was by direct means of the primitive variables.[18,19] The method of stream curvature that is used frequently in the S_2 calculation belongs to this category. In this method, the momentum equation along a certain coordinate is discretized by the two-point difference, and the difference equation is used to compute the relation between the velocity, density, and the values of velocity at the boundary. All gas variables in the whole flowfield can then be obtained from the continuity equation.

D. Mean Streamline and Stream Surface Methods

As an approximate method, the mean streamline method (MSLM)[20–23] is effective in solving the governing equations on the stream surfaces. Its direct problem can be used to calculate the flow variables along the surface of revolution, whereas while solving the inverse problem, the coordinates of the cascade can be determined. In the solution procedure the partial differentials along the circumferential direction of the gas parameters are first computed and the known gas variables on the mean streamline are then expanded by means of a Taylor series. The circumferential distributions of gas variable are obtained.

The extension of the MSLM to the three-dimensional flow is the mean-stream-surface method.[1,24,25] The differences between these two methods are that in the latter, Taylor expansion is conducted along two directions of coordinates and the constraints on the inner and outer annular walls[26] must be taken into account. The solution method of the latter is similar to that of the former and it may be completed in the following way: Taylor expansion is done first along one direction and continued in another direction, and the three-dimensional flow parameters are obtained.

In recent years, Wu's three-dimensional flow theory has been developed further and is applied to many engineering problems in China as well as other countries. In this paper only its theoretical developments and some applications to designing the turbomachinery in China are presented.

IV. Methods for Solving Transonic Flow Along the Stream Surface

Two methods have been developed to solve the transonic flows on stream surfaces. One is the stream function method and the other is the time-dependent method.

A. Transonic Stream Function Formulation

For this method, it is known that there is the double-value problem of the density in the transonic regime when it is computed from the mass flux by a transcendental equation. To circumvent this difficulty, and more importantly, to ensure the satisfaction of the Rankine–Hugoniot condition by the captured shock, one of the momentum equations is adopted as the principal equation of the stream function and another momentum equation is used to compute the density directly; the whole solution process consists of the iterative calculation between the stream function equation and the density equation.[27] Take the flow on a surface of revolution as an example. These two equations then have the following form, respectively:

$$\frac{\partial}{\partial x^2}\left[\frac{1}{\rho}\left(A\frac{\partial\psi}{\partial x^2} - B\frac{\partial\psi}{\partial x^1}\right)\right] - \frac{\partial}{\partial x^1}\left[\frac{1}{\rho}\left(B\frac{\partial\psi}{\partial x^2} - C\frac{\partial\psi}{\partial x^2}\right)\right] = D \qquad (17)$$

$$\begin{aligned}\frac{\partial\rho}{\partial x^1} = {} & -\rho\frac{\partial\,\ell n\,T}{\partial x^1} - \frac{1}{\tau RT}\left\{\frac{\partial}{\partial x^1}\left(\tau\frac{m^2}{\rho}\right) + \cos\theta_{12}\frac{\partial}{\partial x^1}\left(\tau\frac{mn}{\rho}\right)\right. \\ & \left. + \sqrt{\frac{a_{11}}{a_{22}}}\left[\left(\frac{mn}{\rho}\right) + \cos\theta_{12}\frac{\partial}{\partial x^2}\left(\tau\frac{n^2}{\rho}\right)\right]\right\} \\ & + \frac{1}{\rho RT}\left\{(n^2 - m^2)\frac{\partial(\ell n\sqrt{a_{22}})}{\partial x^1} - 2n(m + n\cos\theta_{12})\right. \\ & \times\sqrt{\frac{a_{11}}{a_{22}}}\frac{\partial(\ell n\sqrt{a_{11}})}{\partial x^2} - \sqrt{\frac{a_{11}}{a_{22}}}n^2\frac{\partial\cos\theta_{12}}{\partial x^2} \\ & \times(m + n\cos\theta_{12})\sqrt{a_{11}}\left[\frac{m}{\sqrt{a_{11}}}\frac{\partial(\ell n\sin\theta_{12})}{\partial x^1}\right. \\ & \left.\left. + \frac{n}{\sqrt{a_{22}}}\frac{\partial(\ell n\sin\theta_{12})}{\partial x^2}\right] + 2\rho n\omega\sqrt{a_{11}}\sin\theta_{12} + \rho^2|\omega|^2 r\right\}\end{aligned} \qquad (18)$$

where

$$A_1 = \sqrt{a_{11}}/(\tau\sqrt{a_{22}}\sin\theta_{12}), \qquad B = \cos\theta_{12}/(\tau\sin\theta_{12})$$

$$C = \sqrt{a_{22}}/(\tau\sqrt{a_{11}}\sin\theta_{12})$$

$$D = \frac{1}{w_1}\left(\frac{\partial I}{\partial x^2} - \frac{R}{\kappa - 1}\frac{\partial T}{\partial x^2} + RT\frac{\partial\,\ell n\,\rho}{\partial x^2}\right) + E$$

$$E = 2\omega\sin\sigma\sqrt{a_{11}a_{22}}\sin\theta_{12}, \qquad m = \rho w^1, \qquad n = \rho w^2$$

To solve Eq. (17), which is a mixed-type equation, a type-dependent differencing scheme[28] should be employed: the central difference in the subsonic region and the upwinding scheme for the supersonic part. If the supersonic flow exists only in the x^1 direction, the artificial velocity may be introduced as

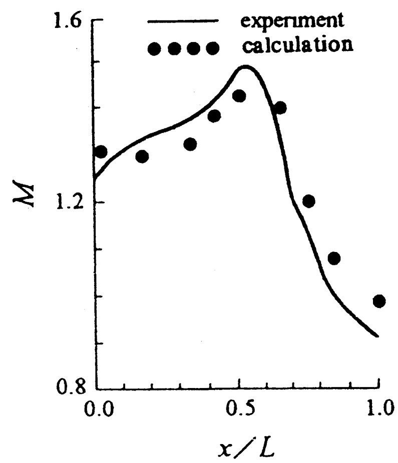

Fig. 7 Mach number distributions for near tip section of NASA Lewis fan.

$$w_{ij}^2 = w_{ij}^2 - \nu(w_{ij}^2 - w_{i-1,j}^2)$$

which is the correct expression for the numerical viscosity.[29] It is also demonstrated that the artificial density[30] introduced in the transonic potential calculations is no longer valid for the transonic stream function computation because the original equations are different. In the potential approach the continuity equation is converted to its principle equation, whereas in the stream function method the continuity equation is automatically satisfied and one of the momentum equations is taken as the principle equation. The principle equation of the stream function, the density equation, and the expression of the artificial viscosity constitutes the complete transonic stream function formulation, which now is an exact mathematical model equivalent to Euler equations in theory, and is as simple and efficient as the potential method in computation. One of the numerical results on the surface of revolution by use of this formulation is shown in Fig. 7.

B. Time-Dependent Approach

In recent years, the time-dependent method to solve the transonic flows has been quickly developed. The method has been applied to computing the flows on the stream surfaces.[31,32] At this time, the governing equations along the surface of revolution in nonorthogonal curvilinear coordinates may be written as

$$\frac{\partial(\rho\tau\sqrt{a})}{\partial t} + \frac{\partial(\rho\tau\sqrt{a}w^1)}{\partial x^1} + \frac{\partial(\rho\tau\sqrt{a}w^2)}{\partial x^2} = 0 \qquad (19)$$

$$\frac{\partial w_1}{\partial t} - w^2\left(\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\right) - 2w^2\omega^3\sqrt{a} = -\left(\frac{\partial I}{\partial x^1} - T\frac{\partial s}{\partial x^1}\right) \qquad (20)$$

$$\frac{\partial w_2}{\partial t} - w^1\left(\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\right) - 2w^1\omega^3\sqrt{a} = -\left(\frac{\partial I}{\partial x^2} - T\frac{\partial s}{\partial x^2}\right) \qquad (21)$$

$$\frac{\partial I}{\partial t} + w^1\frac{\partial I}{\partial x^1} + w^2\frac{\partial I}{\partial x^2} - \frac{1}{\rho}\frac{\partial p}{\partial t} = 0 \qquad (22)$$

An improved version of the MacCormark difference scheme is used to solve Eqs. (19–22) in Ref. 33. [In the actual calculation, Eq. (22) is replaced by the simple relation of I = const]. Figure 8 shows the distribution of the Mach number

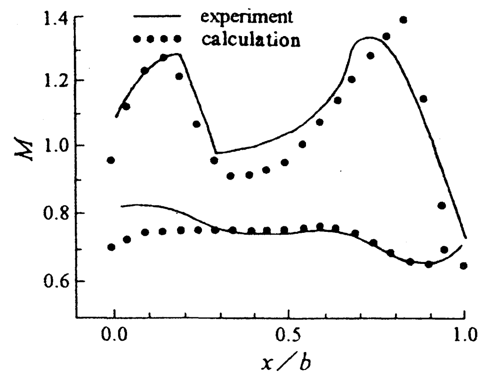

Fig. 8 Mach number distributions of T_1 cascade profile.

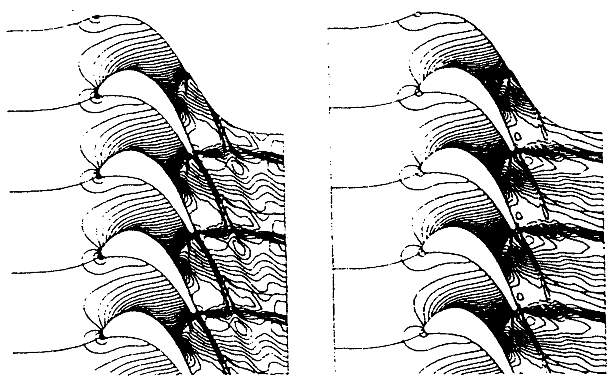

Fig. 9 Iso-Mach and isodensity lines of VK1 turbine cascade.

for the T_1 cascade by this method. Two shocks exist in the cascade channel.

The time-dependent method is more frequently applied when solving Euler equations on the surface of revolution.[34–37] Some high-resolution schemes for capturing the shock are also successfully used in computations. In Fig. 9, the computed iso-Mach lines and isodensity lines in a turbine cascade are plotted to show the capability of the total variation diminishing Beam–Warming scheme to capture the shocks. The complicated shock system near the trailing edge is clearly demonstrated.

It should be pointed out that across a shock the stream surface undergoes an abrupt turning, and that the thickness of the stream filaments becomes discontinuous. The spatiality of the shock makes coupling between the stream surfaces stronger. When the three-dimensional shocked flow is solved by means of an iterative calculation between the two-dimensional transonic flows on two families of stream surfaces, the shape of the stream surface, generally, cannot be specified in advance.[6,8] For the purpose of computing this three-dimensional flow along two kinds of stream surfaces, the steady stream surface is extended to the unsteady one, and the basic equations of motion on two types of the unsteady stream surfaces are derived in the four-dimensional space, and the corresponding boundary and initial conditions suitable to solve the steady problem are obtained from the characteristic compatibility relations.[38] The assumption of the transonic surface of revolution extensively used in the subsonic flow is no longer valid, and a concept of the generalized surface of revolution is proposed.[6,8] Its thickness is a function of both streamwise and tangential coordinates and may be discontinuous. Some calculation of transonic flow on such a generalized surface of revolution is completed by means of the stream function formulation.[27,29]

V. Concluding Remarks

Based on a deep and thorough analysis of the nature of the complex flow in turbomachinery and the characteristic features of its governing equations, three fundamental assumptions were made in the three-dimensional flow theory of turbomachinery. Under these assumptions, the three-dimensional flow is in a blade row decomposed of a series of two-dimensional flows on the two families of stream surfaces through the partial derivatives along these stream surfaces. These derivatives link the three-dimensional flow in the blade channel to the flows on the stream surfaces.

It is seen that in the governing equations on the stream surfaces two meaningful quantities appeared: the thickness of stream filament and the force between stream surfaces. They represent the close connection of the two kinds of stream surfaces and make the governing equations of flow along stream surfaces quite different from these in the common two-dimensional flows. They may be considered as two pillars of the three-dimensional flow theory.

The stream function is used to solve the equations on the stream surfaces. The method has been widely applied to subsonic flow problems. In a transonic flow, the artificial velocity is introduced and the complete transonic stream function formulation has become a reality. The three-dimensional flow theory in subsonic and supersonic turbomachines on the iterative solutions between S_1 and S_2 stream surfaces has been successfully extended to the transonic flow.

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